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NON-LINEAR DIFFUSION PROBLEMS WITH VARIABLE DIFFUSIVITY AND TIME-DEPENDENT FLUX BOUNDARY CONDITIONS

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NOMENCLATURE

C, concentration [kg/m^3];
C_i, initial concentration [kg/m^3];
D, diffusivity [m^2/s];
$\mathscr{D}$, dimensionless diffusivity;
E, relative change of diffusivity;
F, function;
k, coefficient;
L, characteristic length [m];
m, exponent;
T, dimensionless time;
t, time [s];
U, dimensionless concentration;
X, dimensionless corodinate;
x, coordinate [m].

Greek symbols

η, similarity variable;
κ, parameter of nonlinearity.

NON-LINEAR diffusion equations with concentration-dependent diffusion coefficient and corresponding heat conduction equations with temperature-dependent thermal properties arise in a number of physical and engineering problems. Some classes of such problems subject to the second kind boundary conditions (flux boundary conditions) have been studied by a few authors and some analytical solutions are known [1–3]. These analytical procedures, however, are valid only for the problems with constant flux boundary conditions.

The present paper establishes a general method of obtaining exact analytical solutions for a certain class of the non-linear and variable flux type diffusion problems.

THEORETICAL TREATMENT

We are concerned with the non-linear diffusion problem with variable diffusivity in a semi-infinite medium:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D(C)\frac{\partial C}{\partial x}\right], \quad 0 \leqq x < \infty \tag{1}$$

where C is the concentration of the diffusing substance, t the time, x the coordinate and D the diffusivity which is a positive function of the concentration. The initial and boundary conditions considered are:

$$\left.\begin{aligned} C &= C_i \quad &\text{at} \quad t = 0 \\ D\frac{\partial C}{\partial x} &= k \cdot t^{m/2} \quad &\text{at} \quad x = 0 \end{aligned}\right\} \tag{2}$$

where k is an arbitrary constant and m is a positive integer or zero.

Let us introduce the following transformations*

$$\left.\begin{aligned} U &= \frac{D(C_i)^{m/2}}{k \cdot L^{m+1}} \int_C^{C_i} D(C)\,\mathrm{d}C \\ T &= D(C_i)t/L^2 \\ X &= x/L \\ \mathscr{D} &= D(C)/D(C_i) \end{aligned}\right\} \tag{3}$$

where, functional form of the dimensionless diffusivity $\mathscr{D}(U)$ depends directly on that of the original diffusivity $D(C)$. In most cases of practical importance, however, it has been shown that the dimensionless diffusivity can be represented by a simple polynomial series of U and a certain parameter κ [3]. When the diffusivity $D(C)$, for instance, is described by an exponential function, the dimensionless diffusivity $\mathscr{D}$ then becomes:

$$\mathscr{D}(U) = 1 - \kappa \cdot U \tag{4}$$

where the non-linearity parameter κ is defined by:

$$\left.\begin{aligned} \kappa &= \frac{k \cdot L^{m+1}}{C_i \cdot D(C_i)^{m/2+1}} \ln E \\ E &= D(C_i)/D(C=0). \end{aligned}\right\} \tag{5}$$

Equations (1) and (2) then become:

$$\left.\begin{aligned} &\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2} - \kappa \cdot U\frac{\partial^2 U}{\partial X^2} \\ &U = 0 \quad \text{at} \quad T = 0 \\ &\frac{\partial U}{\partial X} = -T^{m/2} \quad \text{at} \quad X = 0. \end{aligned}\right\} \tag{6}$$

The perturbation solution for this non-linear equation is described as:

$$U = U_1 + \kappa \cdot U_2 + \kappa^2 \cdot U_3 + \ldots. \tag{7}$$

Applying the similarity analysis and the group invariant theory [3], one can derive that:

$$\left.\begin{aligned} U_j(T, X) &= T^{(m+1)j/2} \cdot F_j(\eta), \quad j = 1,2,3,\ldots \\ \eta &= X/2\sqrt{T}. \end{aligned}\right\} \tag{8}$$

Substituting equations (8) and (7) into (6) and collecting coefficients of like powers of the parameter κ, one can obtain the following simultaneous ordinary differential equations for the unknown similarity functions F_j as:

$$\left.\begin{aligned} &F_1'' + 2\eta F_1' - 2(m+1)F_1 = 0 \\ &F_2'' + 2\eta F_2' - 4(m+1)F_2 = F_1 F_1'' \\ &F_3'' + 2\eta F_3' - 6(m+1)F_3 = F_1 F_2'' + F_2 F_1''. \end{aligned}\right\} \tag{9}$$

*Where, L is a characteristic length, unit length for instance.

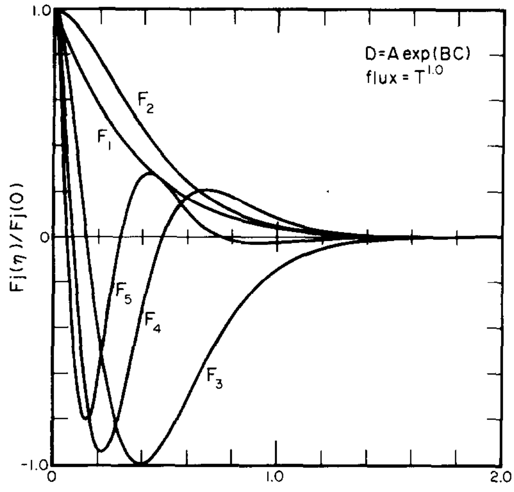

FIG. 1. Similarity functions F_j.

The boundary conditions for these differential equations are derived from the conditions in equation (6) as:

$$\left.\begin{aligned} &F_1'(0) = -2 \\ &F_2'(0) = F_3'(0) = F_4'(0) = \ldots = 0 \\ &F_1(\infty) = F_2(\infty) = F_3(\infty) = \ldots = 0. \end{aligned}\right\} \quad (10)$$

This system of the two point boundary value problems of the simultaneous linear ordinary differential equations can be easily solved by an analytical or numerical method. The dimensionless concentration U, then can be evaluated by making use of equation (8) and (7). The concentration distribution $C(t, x)$ can be evaluated by the inverse of the transformation (3). When the attention is focused on the change of the concentration at the surface ($x = 0$), we need the numerical values of F_j only at the origin ($\eta = 0$).

Some examples of the similarity functions F_j and corresponding calculated results of the surface concentration changes are shown in Fig. 1, Table 1 and Fig. 2.

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Table 1. Numerical values of $F_j(0)$ for the exponential type diffusivities

	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
$F_1(0)$	1.1283791671	0.8862269520	0.7522525787	0.6646701941	0.6018022225	0.5538917779
$F_2(0)$	−0.25	−0.1575405335	−0.1145832367	−0.0899224361	−0.0739582886	−0.0627921378
$F_3(0)$	−0.0168406518	−0.0076732703	−0.0045365178	−0.0030651348	−0.0022438633	−0.0017324060
$F_4(0)$	−0.0042613641	−0.0014848700	−0.0007384711	−0.0004391605	−0.0002905108	−0.0002062057
$F_5(0)$	−0.0015019148	−0.0004036669	−0.0001693527	−0.0000887240	−0.0000530505	−0.0000346195
$F_6(0)$	−0.0006244172	−0.0001296251	−0.0000458750	−0.0000211686	−0.0000114382	−0.0000068613
$F_7(0)$	−0.0002871451	−0.0000460708	−0.0000137543	−0.0000055898	−0.0000027293	−0.0000015049
$F_8(0)$	−0.0001414260	−0.0000175444	−0.0000044186	−0.0000015815	−0.0000006978	−0.0000003536
$F_9(0)$	−0.0000732197	−0.0000070249	−0.0000014926	−0.0000004705	−0.0000001876	−0.0000000874
$F_{10}(0)$	−0.0000393752	−0.0000029223	−0.0000005238	−0.0000001454	−0.0000000524	−0.0000000224
$F_{11}(0)$	−0.0000218176			−0.0000000463	−0.0000000151	−0.0000000059
$F_{12}(0)$	−0.0000123848			−0.0000000151	−0.0000000044	−0.0000000016
$F_{13}(0)$	−0.0000071720					
$F_{14}(0)$	−0.0000042234					
$F_{15}(0)$	−0.0000025228					

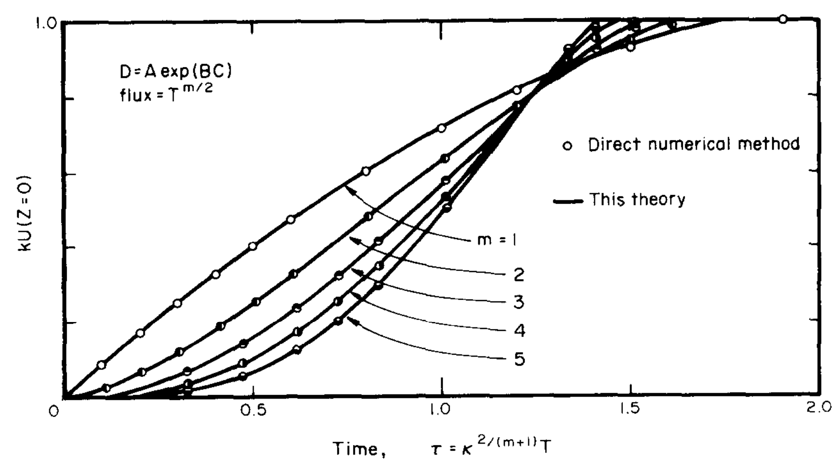

FIG. 2. Change of the surface concentration.